Lecture 8

In this lecture we begin a study of certain important or "special" functions.

Some special functions

Absolute value

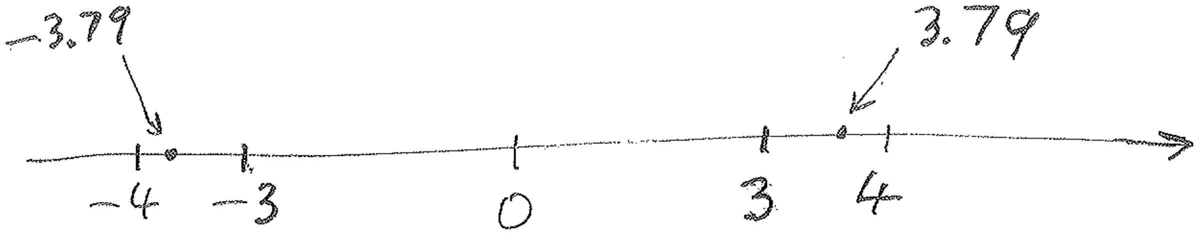
$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

E.g., $|-3| = -(-3)$
$\quad\quad\quad\;\; = 3$

Floor and ceiling

$\lfloor x \rfloor$ = floor of x
$\quad\;\;$ = greatest integer $\leq x$

E.g., $\lfloor 3.79 \rfloor = 3$
$\quad\quad\;\, \lfloor -3.79 \rfloor = -4$



$\lceil x \rceil$ = ceiling of x
= least integer $\geq x$

E.g., $\lceil 3.79 \rceil = 4$

$\lceil -3.79 \rceil = -3$

The absolute value, floor and ceiling functions can be applied to any real number. So their domain is $\mathbb{R}$.

<u>Factorial function</u>

This exists for every integer ≥ 0 (the non-negative integers).

If $n \geq 1$,

$$n! = n(n-1) \cdots 3 \cdot 2 \cdot 1.$$

When $n = 0$,

$$0! = 1 \quad \text{(special definition).}$$

Then we can create a partial table:

n	$n!$
0	1
1	1
2	2
3	6
4	24
5	120
$\vdots$	$\vdots$

$$\boxed{n! = n \cdot (n-1)!} \quad \text{for } n \geq 1$$

This is an example of a <u>recurrence relation</u>.

Some algorithms take "factorial time" or "factorial memory space" to be implemented.

These can be compared with other algorithms that take "polynomial" time or space, "exponential" time or space, or "logarithmic" time or space.

This is part of the study of algorithmic complexity, which is the basis of program feasibility or practicality.

Exponential Functions

An <u>exponential</u> function has the form

$$y = a^x$$

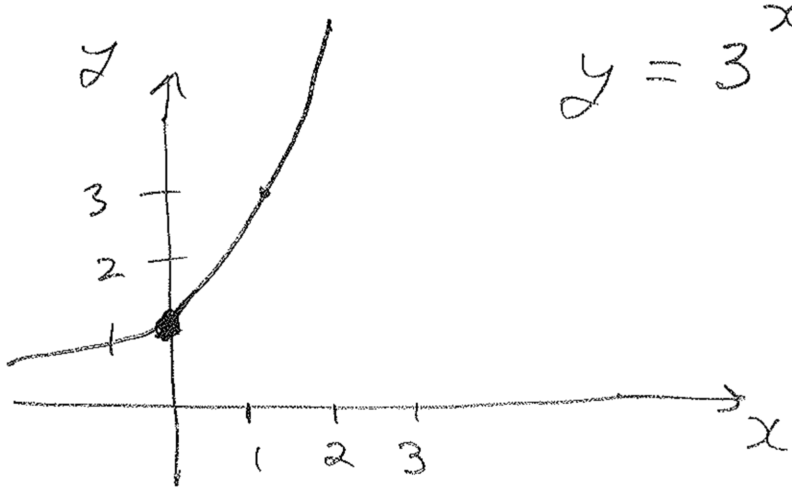
for some constant $a > 0$.

We call a the <u>base</u> and x the <u>exponent</u>. Then $y = a^x$ is a <u>power</u> of a.

For $a > 1$, an exponential function grows very rapidly.

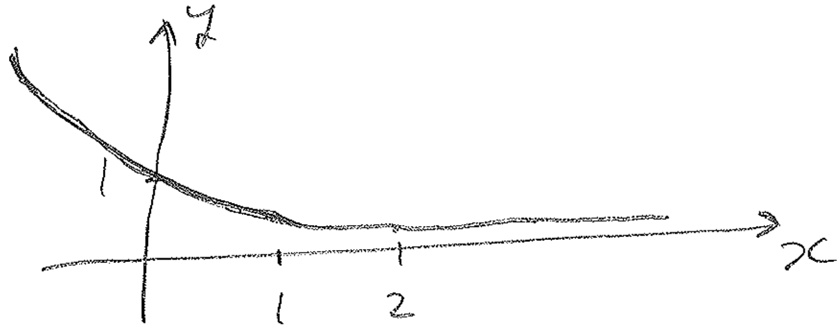
So algorithms or programs that take exponential time or space can't practically be implemented except for small values of the parameters.

$$y = 3^x$$



On the other hand, if $a < 1$ we have a function that models radioactive decay.

$$y = (1/2)^x$$



Laws of exponents

$$a^x a^y = a^{x+y}$$

$$a^x / a^y = a^{x-y}$$

$$a^0 = 1$$

$$a^{-x} = 1/a^x$$

$$(a^x)^y = a^{xy}$$

For $a > 1$,

as $x \to \infty$, $y \to \infty$ (rapidly)

and as $x \to -\infty$, $y \to 0$.

For $a < 1$,

as $x \to \infty$, $y \to 0$

and as $x \to -\infty$, $y \to \infty$.

Logarithmic Functions

We write

$$y = \log_a x$$

to mean that

$$a^y = x.$$

Here, y is the "log of x, to the base a".

<u>E.g.</u>

$$\log_5 25 = \boxed{} \; ?$$

<u>Sol^n</u>

$$5^{\boxed{}} = 25 \; ?$$

What goes in the box? 2

$$5^2 = 25$$

$$\therefore \log_5 25 = 2$$

<u>E.g.</u>

$$\log_3 81 = \boxed{} \; ?$$

<u>Sol^n</u>

$$3^{\boxed{}} = 81 \; ?$$

The no. 4 goes in the box.

$$3^4 = 81$$

$$\therefore \log_3 81 = 4.$$